On Strongly Regular Graphs, Friendship, and the Shannon Capacity

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Graph Spectrum

Throughout this presentation,

- $\mathsf{G} = (\mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G}))$ is a finite, undirected, and simple graph of order $|\mathsf{V}(\mathsf{G})| = n$ and size $|\mathsf{E}(\mathsf{G})| = m$.
- $\mathbf{A} = \mathbf{A}(\mathsf{G})$ is the *adjacency matrix* of the graph.
- The eigenvalues of $\mathbf{A}$ are given in decreasing order by

$$\lambda_{\max}(\mathsf{G}) = \lambda_1(\mathsf{G}) \geq \lambda_2(\mathsf{G}) \geq \ldots \geq \lambda_n(\mathsf{G}) = \lambda_{\min}(\mathsf{G}). \tag{1.1}$$

- The *spectrum* of G is a multiset that consists of all the eigenvalues of $\mathbf{A}$, including their multiplicities.

Orthogonal Representation of Graphs

Definition 1.1

Let G be a finite, undirected and simple graph.
An orthogonal representation of G in $\mathbb{R}^d$

$$i \in \mathsf{V}(\mathsf{G}) \;\mapsto\; \mathbf{u}_i \in \mathbb{R}^d$$

such that

$$\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j = 0, \quad \forall \, \{i, j\} \notin \mathsf{E}(\mathsf{G}).$$

An orthonormal representation of G: $\|\mathbf{u}_i\| = 1$ for all $i \in \mathsf{V}(\mathsf{G})$.

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In an orthogonal representation of a graph G:

- non-adjacent vertices: mapped to orthogonal vectors;
- adjacent vertices: not necessarily mapped to non-orthogonal vectors.

Lovász ϑ-function

Let G be a finite, undirected and simple graph.

The Lovász ϑ-function of G is defined as

$$\vartheta(\mathsf{G}) \triangleq \min_{\mathbf{u},\mathbf{c}} \max_{i \in \mathsf{V}(\mathsf{G})} \frac{1}{\left(\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right)^2}, \tag{1.2}$$

where the minimum is taken over

- all orthonormal representations $\{\mathbf{u}_i : i \in \mathsf{V}(\mathsf{G})\}$ of G, and
- all unit vectors $\mathbf{c}$.

The unit vector $\mathbf{c}$ is called the *handle* of the orthonormal representation.

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$$\left|\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right| \leq \|\mathbf{c}\| \, \|\mathbf{u}_i\| = 1 \implies \vartheta(\mathsf{G}) \geq 1,$$

with equality if and only if G is a complete graph.

An Orthonormal Representation of a Pentagon

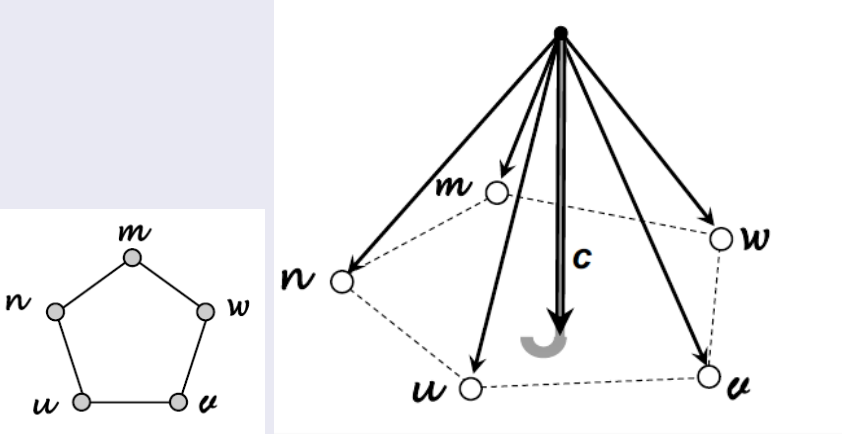

Figure 1: A 5-cycle graph and its orthonormal representation (also known as Lovász umbrella). Calculation shows that $\vartheta(\mathsf{C}_5) = \sqrt{5}$ (Lovász, 1979).

Lovász ϑ-function (Cont.)

- $\mathbf{A}$ is the $n \times n$ adjacency matrix of G $\quad (n \triangleq |\mathsf{V}(\mathsf{G})|)$;
- $\mathbf{J}_n$ is the all-ones $n \times n$ matrix;
- $\mathcal{S}_+^n$ is the set of all $n \times n$ positive semidefinite matrices.

Semidefinite program (SDP), with strong duality, for computing $\vartheta(\mathsf{G})$:

$$\boxed{\begin{array}{l} \text{maximize} \;\; \mathrm{Trace}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \in \mathcal{S}_+^n, \;\; \mathrm{Trace}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \;\Rightarrow\; B_{i,j} = 0, \quad i,j \in [n]. \end{cases} \end{array}}$$

Computational complexity: $\exists$ algorithm (based on the ellipsoid method) that numerically computes $\vartheta(\mathsf{G})$, for every graph G, with precision of r decimal digits, and polynomial-time in n and r.

Lovász ϑ-function (Cont.)

Let $\alpha(\mathsf{G})$, $\omega(\mathsf{G})$, and $\chi(\mathsf{G})$ denote the independence number, clique number, and chromatic number of a graph G. Then,

1. Sandwich theorem:

$$\alpha(\mathsf{G}) \leq \vartheta(\mathsf{G}) \leq \chi(\overline{\mathsf{G}}), \tag{1.3}$$

$$\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi(\mathsf{G}). \tag{1.4}$$

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2. Computational complexity:
   - $\alpha(\mathsf{G})$, $\omega(\mathsf{G})$, and $\chi(\mathsf{G})$ are NP-hard problems.
   - However, the numerical computation of $\vartheta(\mathsf{G})$ is in general feasible by convex optimization (SDP problem).

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2. Computational complexity:
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   - However, the numerical computation of $\vartheta(\mathsf{G})$ is in general feasible by convex optimization (SDP problem).

3. Hoffman-Lovász inequality: Let G be d-regular of order n. Then,
$$\vartheta(\mathsf{G}) \le -\frac{n\,\lambda_n(\mathsf{G})}{d-\lambda_n(\mathsf{G})}, \tag{1.5}$$
with equality if G is edge-transitive.

Strongly Regular Graphs

Let G be a d-regular graph of order n. It is a *strongly regular* graph (SRG) if there exist nonnegative integers λ and μ such that

- Every pair of adjacent vertices have exactly λ common neighbors;
- Every pair of distinct and non-adjacent vertices have exactly μ common neighbors.

Such a strongly regular graph is denoted by srg(n, d, λ, μ).

Theorem 1.2 (Bounds on Lovász function of Regular Graphs, I.S., '23)

Let G be a d-regular graph of order n, which is a non-complete and non-empty graph. Then, the following bounds hold for the Lovász ϑ-function of G and its complement $\overline{\mathsf{G}}$:

1)

$$\frac{n-d+\lambda_2(\mathsf{G})}{1+\lambda_2(\mathsf{G})} \leq \vartheta(\mathsf{G}) \leq -\frac{n\lambda_n(\mathsf{G})}{d-\lambda_n(\mathsf{G})}. \tag{1.6}$$

- Equality holds in the leftmost inequality if $\overline{\mathsf{G}}$ is both vertex-transitive and edge-transitive, or if G is a strongly regular graph;
- Equality holds in the rightmost inequality if G is edge-transitive, or if G is a strongly regular graph.

Cont. of Theorem 1.2

2)

$$1 - \frac{d}{\lambda_n(\mathsf{G})} \le \vartheta(\overline{\mathsf{G}}) \le \frac{n\bigl(1+\lambda_2(\mathsf{G})\bigr)}{n-d+\lambda_2(\mathsf{G})}. \tag{1.7}$$

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A Common Sufficient Condition

All inequalities hold with equality if G is strongly regular. (Recall that the graph G is strongly regular if and only if $\overline{\mathsf{G}}$ is so).

Lovász Function of Strongly Regular Graphs (I.S., '23)

Let G be a strongly regular graph with parameters $\mathrm{srg}(n, d, \lambda, \mu)$. Then,

$$\vartheta(\mathsf{G}) = \frac{n\,(t + \mu - \lambda)}{2d + t + \mu - \lambda}, \tag{1.8}$$

$$\vartheta(\overline{\mathsf{G}}) = 1 + \frac{2d}{t + \mu - \lambda}, \tag{1.9}$$

where

$$t \triangleq \sqrt{(\mu - \lambda)^2 + 4(d - \mu)}. \tag{1.10}$$

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New Relation for Strongly Regular Graphs

$$\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) = n, \tag{1.11}$$

holding not only for all vertex-transitive graphs (Lovász '79), but also for all strongly regular graphs (that are not necessarily vertex-transitive).

We next provide an original proof of the following celebrated theorem by Erdös, Rényi and Sós (1966), based on our expression for the Lovász ϑ-function of strongly regular graphs (and their complements, which are also strongly regular graphs).

Theorem 1.3 (Friendship Theorem)

Let G be a finite graph in which any two distinct vertices have a single common neighbor. Then, G has a vertex that is adjacent to every other vertex.

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Let G be a finite graph in which any two distinct vertices have a single common neighbor. Then, G has a vertex that is adjacent to every other vertex.

A Human Interpretation of Theorem 1.3

- There is a party with n people, where every two people have precisely one common friend in that party.
- Theorem 1.3 asserts that one of these people is everybody's friend.
- Indeed, construct a graph whose vertices represent the n people, and every two vertices are adjacent if and only if they represent two friends. The claim then follows from Theorem 1.3.

Remark 1 (On the Friendship Theorem - Theorem 1.3)

- The windmill graph (see Figure 2) has the desired property, and it turns out to be the only one graph with that property.
- The friendship theorem does not hold for infinite graphs.

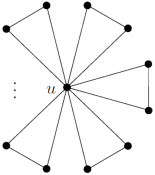


Figure 2: Windmill graph.

Alternative Proof of Theorem 1.3 (I.S., '25)

Suppose the assertion is false, and G is a counterexample — a finite graph in which any two distinct vertices have a single common neighbor, yet no vertex in G is adjacent to all other vertices. A contradiction is obtained by the following proof outline:

- It is shown that the graph is regular.
- It is then shown that the graph is strongly regular srg$(n, k, 1, 1)$.
- If $k = 0$ or $k = 2$, then $\mathsf{G} = \mathsf{K}_1$ or $\mathsf{G} = \mathsf{K}_3$, respectively, which satisfy the assertion of the theorem. Hence, next assume that $k \geq 3$.
- By the theorem hypothesis, it follows that $\omega(\mathsf{G}) = \chi(\mathsf{G}) = 3$.
- By the sandwich theorem $\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi(\mathsf{G})$, so $\vartheta(\overline{\mathsf{G}}) = 3$.
- Based on the expression for the Lovász ϑ-function $\vartheta(\overline{\mathsf{G}}) = 1 + \frac{k}{\sqrt{k-1}}$.
- This leads to a contradiction for all $k \geq 3$.

The sandwich theorem for the Lovász ϑ-function applied to strongly regular graphs gives the following result.

Corollary 1.4 (Bounds on Parameters of SRGs)

Let G be a strongly regular graph with parameters srg(n, d, λ, μ). Then,

$$\alpha(\mathsf{G}) \leq \left\lfloor \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rfloor \tag{1.12}$$

$$\omega(\mathsf{G}) \leq 1 + \left\lfloor \frac{2d}{t+\mu-\lambda} \right\rfloor, \tag{1.13}$$

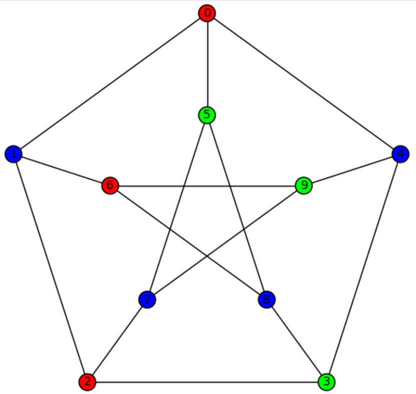
$$\chi(\mathsf{G}) \geq 1 + \left\lceil \frac{2d}{t+\mu-\lambda} \right\rceil, \tag{1.14}$$

$$\chi(\overline{\mathsf{G}}) \geq \left\lceil \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rceil, \tag{1.15}$$

with

$$t \triangleq \sqrt{(\mu-\lambda)^2 + 4(d-\mu)}. \tag{1.16}$$

Examples: Bounds on Parameters of SRGs



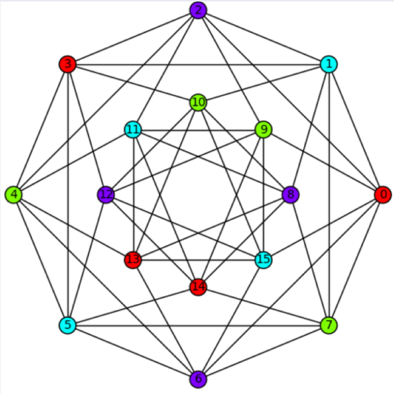


Figure 3: The Petersen graph is srg$(10, 3, 0, 1)$ (left), and the Shrikhande graph is srg$(16, 6, 2, 2)$ (right). Their chromatic numbers are 3 and 4, respectively.

Schläfli Graph

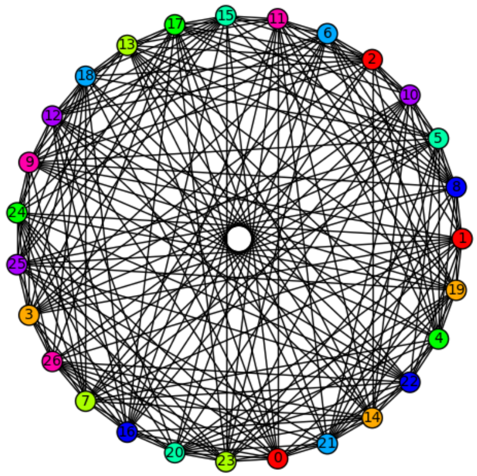


Figure 4: Schläfli graph is $\mathrm{srg}(27, 16, 10, 8)$ with chromatic number $\chi(\mathsf{G}) = 9$.

Examples: Bounds on Parameters of SRGs (Cont.)

1. Let G_1 be the Petersen graph. Then, the bounds on the independence, clique, and chromatic numbers of G are tight:
$$\alpha(\mathsf{G}_1) = 4, \quad \omega(\mathsf{G}_1) = 2, \quad \chi(\mathsf{G}_1) = 3. \tag{1.17}$$
2. The bounds on the chromatic numbers of the Schläfli graph (G_2), Shrikhande graph (G_3) and Hall-Janko graph (G_4) are tight:
$$\chi(\mathsf{G}_2) = 9, \quad \chi(\mathsf{G}_3) = 4, \quad \chi(\mathsf{G}_4) = 10. \tag{1.18}$$
3. For the Shrikhande graph (G_3),
   - the bound on its independence number is also tight: $\alpha(\mathsf{G}_3) = 4$,
   - its upper bound on its clique number is, however, not tight (it is equal to 4, and $\omega(\mathsf{G}_3) = 3$).

Strong Product of Graphs

Let G and H be two graphs. The strong product $\mathsf{G} \boxtimes \mathsf{H}$ is a graph with

- vertex set: $\mathsf{V}(\mathsf{G} \boxtimes \mathsf{H}) = \mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{H})$,
- two distinct vertices (g, h) and (g', h') in $\mathsf{G} \boxtimes \mathsf{H}$ are adjacent if the following two conditions hold:
  1. $g = g'$ or $\{g, g'\} \in \mathsf{E}(\mathsf{G})$,
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Strong products are commutative and associative.

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Strong Powers of Graphs

Let

$$\mathsf{G}^{\boxtimes k} \triangleq \underbrace{\mathsf{G} \boxtimes \ldots \boxtimes \mathsf{G}}_{\mathsf{G} \text{ appears } k \text{ times}}, \quad k \in \mathbb{N} \tag{1.19}$$

denote the k-fold strong power of a graph G.

Shannon Capacity of a Graph (1956)

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$$\begin{aligned}\Theta(\mathsf{G}) &= \sup_{k\in\mathbb{N}} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})} \\ &= \lim_{k\to\infty} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})}. \qquad (2.1)\end{aligned}$$

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- The last equality holds by Fekete's Lemma since the sequence $\{\log \alpha(\mathsf{G}^{\boxtimes k})\}_{k=1}^{\infty}$ is super-additive, i.e.,

$$\alpha(\mathsf{G}^{\boxtimes (k_1+k_2)}) \geq \alpha(\mathsf{G}^{\boxtimes k_1})\, \alpha(\mathsf{G}^{\boxtimes k_2}). \tag{2.2}$$

On the Computability of the Shannon Capacity of Graphs

- The Shannon capacity of a graph can be rarely computed exactly. ☹
- However, the Lovász ϑ-function of a graph is a computable (and sometimes tight) upper bound on the Shannon capacity. ☺

Lovász Bound on the Shannon Capacity of Graphs (1979)

Theorem: For every finite, simple and undirected graph G,

$$\alpha(\mathsf{G}) \leq \Theta(\mathsf{G}) \leq \vartheta(\mathsf{G}), \tag{2.3}$$

so if $\alpha(\mathsf{G}) = \vartheta(\mathsf{G})$, then $\Theta(\mathsf{G}) = \vartheta(\mathsf{G})$.

Shannon Capacities of Some Strongly Regular Graphs

1. The Hall-Janko graph G is srg$(100, 36, 14, 12)$, and $\Theta(\mathsf{G}) = 10$.
2. The Hoffman-Singleton graph G is srg$(50, 7, 0, 1)$, and $\Theta(\mathsf{G}) = 15$.
3. The Janko-Kharaghani graphs of orders 936 and 1800 are srg$(936, 375, 150, 150)$ and srg$(1800, 1029, 588, 588)$, respectively. The capacity of both graphs is 36.
4. Janko-Kharaghani-Tonchev: $\mathsf{G} =$ srg$(324, 153, 72, 72), \Theta(\mathsf{G}) = 18$.
5. The graphs introduced by Makhnev are $\mathsf{G} =$ srg$(64, 18, 2, 6)$ and $\overline{\mathsf{G}} =$ srg$(64, 45, 32, 30)$. Capacities: $\Theta(\mathsf{G}) = 16$, and $\Theta(\overline{\mathsf{G}}) = 4$.
6. The Mathon-Rosa graph G is srg$(280, 117, 44, 52)$, and $\Theta(\mathsf{G}) = 28$.
7. The Schläfli graph G is srg$(27, 16, 10, 8)$, and $\Theta(\mathsf{G}) = 3$.
8. The Shrikhande graph is srg$(16, 6, 2, 2)$; its capacity is $\Theta(\mathsf{G}) = 4$.
9. The Sims-Gewirtz graph G is srg$(56, 10, 0, 2)$, and $\Theta(\mathsf{G}) = 16$.
10. The graph G by Tonchev is srg$(220, 84, 38, 28)$, and $\Theta(\mathsf{G}) = 10$.

Theoretical results on the Shannon capacity of graphs are in the papers.

Recent Journal Papers

This talk relies on the following recent journal papers:

1. I. Sason, "Observations on the Lovász ϑ-function, graph capacity, eigenvalues, and strong products," *Entropy*, vol. 25, no. 1, paper 104, pp. 1–40, January 2023. https://doi.org/10.3390/e25010104
2. I. Sason, "Observations on graph invariants with the Lovász ϑ-function," *AIMS Mathematics*, vol. 9, pp. 15385–15468, April 2024. https://www.aimspress.com/article/doi/10.3934/math.2024747
3. I. Sason, "On strongly regular graphs and the friendship theorem," submitted, February 2025. https://arxiv.org/abs/2502.13596